Subgroup Theorems for the Baer-invariant of Groups

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Abstract

M.R.Jones and J.Wiegold in [3] have shown that if G is a finite group with a subgroup H of finite index n , then the n-th power of Schur multiplier of G , $M(G)^n$, is isomorphic to a subgroup of $M(H)$.

In this paper we prove a similar result for the centre by centre by w variety of groups, where w is any outer commutator word. Then using a result of M.R.R.Moghaddam [6], we will be able to deduce a result of Schur's type (see [4,9]) with respect to the variety of nilpotent groups of class at most c ($c \geq 1$), when $c + 1$ is any prime number or 4.

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1. INTRODUCTION

Let F_∞ be the free group freely generated by an infinite set $\{x_1, x_2, \ldots\}$. Let w be an outer commutator word in F_∞, and assume $\mathcal{V}$ to be the variety of groups defined by the commutator word $v = [w, x_1, x_2]$. Let G be a finite group with a $\mathcal{V}$- stem cover $G^*/L \cong G$, say. If H is a subgroup of G of finite index n such that $H \cong B/L$, then it is shown (Theorem 3.6) that the commutator subgroup $[w(G), \mathcal{V}M(G)^n]$ is isomorphic to a subgroup of $\mathcal{V}M(H)$, where $w(G)$ is the verbal subgroup of G with respect to the word w, and $\mathcal{V}M(G)$ is the Baer-invariant of G with respect to the variety of groups $\mathcal{V}$.

Now, if we assume w to be an empty word, then we obtain the result of M.R.Jones and J.Wiegold [3]. To prove the above fact, one needs to have the concept of the transfer of a group (see [8]) in the above variety $\mathcal{V}$, which has been done in the Definition 3.3, (see also Proposition 3.4).

I.Schur in 1904 [9] showed that if G is a finite group and G_p is a Sylow p-subgroup of G, then the Sylow p-subgroup of the Schur multiplier of G, denoted by $M(G)_p$, is

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isomorphic to a subgroup of the Schur multiplier of its Sylow p-subgroup, $M(G_p)$.

In section 4, by using our previous result together with a result of M.R.R.Moghaddam (see Theorem 2.3 [6]), it is proved a theorem of Schur's type (Theorem 4.3) with respect to the variety of nilpotent groups, $\mathcal{N}_c$, of class at most c , where $c + 1$ is any prime number or $c = 3$.

Finally, in section 5 we give an example to show that the main result can not be futher generalized. In other words, it is in best possible shape.

2. NOTATION AND PRELIMINARIES

Let x, y be two elements of a group G then $[x, y]$, the *commutator* of x and y , and x^y denote the elements $x^{-1}y^{-1}xy$ and $y^{-1}xy$, respectively. The commutator of higher weight is defined inductively as follows:

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] \ .$$

If H and K are two subgroups of a group G then $[H, K]$ denotes the subgroup of G generated by all the commutators $[h, k]$ with h in H and k in K . In particular, if $H = K = G$ then $[G, G]$, which is denoted by G' , is the *derived subgroup* of G . The lower and upper centeral series are denoted by $\gamma_c(G) \ (c \geq 1)$ and $Z_d(G) \ (d \geq 0)$, respectively.

Let F_∞ be the free group freely generated by an infinite countable set $\{x_1, x_2, \ldots\}$. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_1, \ldots, x_s)$ are two words in F_∞ , then the *composite*

3

of u and v , $u \circ v$, is defined as follows:

$$u \circ v = u(v(x_1, \ldots, x_t), \ldots, v(x_{(s-1)t+1}, \ldots, x_{st})) \ .$$

In particular, the composite of some nilpotent words is called *polynilpotent word,* i.e.

$$\gamma_{c_1+1,\ldots,c_t+1} = \gamma_{c_1+1} \circ \gamma_{c_2+1} \circ \cdots \circ \gamma_{c_t+1} \ ,$$

where γ_{c_i+1} $(1 \le i \le t)$ is a nilpotent word in distinct variables.

Outer commutator words are defined inductively, as follows:

The word x_i is an outer commutator word (henceforth o.c.word) of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are o.c.words of weights s and t , respectively, then

$$w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$$

is an o.c.word of weight $s + t$.

Let V be a subset of F_∞ and $\mathcal{V}$ be the variety of groups defined by the set of laws V . Then following P.Hall [2] in 1957 , introduce the notion of the *verbal subgroup,* $V(G)$, and the *marginal subgroup,* $V^*(G)$, associated with the variety $\mathcal{V}$ and the group G, which we assume that the reader is familiar with these concepts.

If G is a group with a free presentation $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$, then the *Baer_invariant* of G , with respect to the variety $\mathcal{V}$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} \ ,$$

4

where $V(F)$ is the verbal subgroup of F and

$$[RV^*F] =< v(f_1, \ldots, f_ir, \ldots, f_s)v(f_1, \ldots, f_s)^{-1} \mid v \in V, f_i \in F, r \in R, 1 \leq i \leq s > .$$

One notes that the Baer-invariant of the group G is always abelian and independent of the choice of the free presentation of G (see M.R.R.Moghaddam [6] or C.R.Leedham-Green and S.McKay [5]).

In particular, if $\mathcal{V}$ is the variety of abelian or nilpotent groups of class at most c $(c \geq 2)$, then the Baer-invariant of G will be $R \cap F'/[R, F]$, which is isomorphic to the Schur-multiplier of G (I.Schur [9]), or $R \cap \gamma_{c+1}(F)/[R,_c F]$, respectively (M.R.R.Moghaddam [6]).

Let $\mathcal{V}$ be a variety of groups and G be an arbitrary group, then an extension

$$1 \longrightarrow L \longrightarrow E \longrightarrow G \longrightarrow 1 \qquad (*)$$

is called a $\mathcal{V}$-*stem extension* of G , if $L \subseteq V(E) \cap V^*(E)$. The extension $(*)$ will be called $\mathcal{V}$-*stem cover* of G , when we also have $L \cong \mathcal{V}M(G)$, and in this case E is said to be a $\mathcal{V}$-*covering group* of G (see [5]).

Definition 2.1

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V . Then $\mathcal{V}$ is said to be a *Schur-Baer variety*; whenever, the marginal factor group $G/V^*(G)$ is finite of order m , say, then the verbal subgroup $V(G)$ is also finite and $|V(G)| \mid m^k$, $k \in \mathbf{N}$, for arbitrary group G .

Among many other results, I.Schur [9] showed that the variety of abelian groups

has Schur-Baer property. Also R.Baer [1] proved that the variety defined by o.c. words has the same property.

The following results are needed later.

Lemma 2.2

Let $\mathcal{V}$ be a variety defined by the set of laws V , then the following conditions are equvalent:

(i) $\mathcal{V}$ is a Schur-Baer variety.

(ii) For any finite group G , the Baer-invariant $\mathcal{V}$M(G) is of order dividing a power of $|G|$.

Proof. See [5] .

Theorem 2.3 (M.R.R.Moghaddam [6])

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c ,and let G_1 and G_2 be two arbitrary groups. Then

(i) $\mathcal{N}_c M(G_1 \times G_2) \cong \mathcal{N}_c M(G_1) \oplus \mathcal{N}_c M(G_2) \oplus T(G_2, G_1)_{c+1}$, when $c+1$ is a prime number.

(ii) $\mathcal{N}_3 M(G_1 \times G_2) \cong \mathcal{N}_3 M(G_1) \oplus \mathcal{N}_3 M(G_2) \oplus \hat{T}(G_2, G_1)_4$,

where $T(G_2, G_1)_{c+1}$ denotes the summation of all tensor product corresponding to the basic commutators of weight $c+1$ on two letters, and

$$\hat{T}(G_2, G_1)_4 = (\overline{G_2} \otimes \overline{G_1} \otimes \overline{G_1} \otimes \overline{G_1}) \oplus \sum_{i=1}^{2}(\overline{G_2} \otimes \overline{G_1} \otimes \overline{G_i} \otimes \overline{G_1}) \oplus (\overline{G_2} \otimes \overline{G_1}) \wedge (\overline{G_2} \otimes \overline{G_1}) ,$$

where $\overline{G}_i = (G_i)_{ab}$, $i = 1, 2$, and $\wedge$ is the wedge product.

Proof. See [6] .

3. THE MAIN RESULTS

In 1973 , M.R.Jones and J.Wiegold [3] showed that if G is a finite group, and H a subgroup of G of index n , then the n-th power of $M(G)$, i.e. $M(G)^n$, is isomorphic to a subgroup of $M(H)$, which generalizes Schur's Theorem [4,9].

In this section the above result will be generalized to the centre by centre by w variety of groups, where w is any outer commutator word.

We assume G to be a finite group, unless otherwise stated.

Lemma 3.1

Let $\mathcal{V}$ be a variety of groups, and H be a group with a $\mathcal{V}$-marginal subgroup A , i.e. $A \leq V^*(H)$. If $G = H/A$, then $V(H) \cap A$ is a homomorphic image of $\mathcal{V}M(G)$.

Proof.

Let $H \cong F/R$ be a free presentation of H , then for some subgroup T of F , we have $A \cong T/R$, hence $G \cong F/T$, and by the definition

$$\mathcal{V}M(G) = \frac{T \cap V(F)}{[TV^*F]} \quad,$$

and using Dedekind's law,

$$V(H) \cap A = \frac{V(F)R}{R} \cap \frac{T}{R} = \frac{V(F)R \cap T}{R} = \frac{(V(F) \cap T)R}{R}$$

$$\cong \frac{V(F) \cap T}{(V(F) \cap T) \cap R} = \frac{V(F) \cap T}{V(F) \cap R} \quad.$$

Since T/R is a marginal subgroup of F/R , it follows that $[TV^*F] \subseteq R$. So we have

$$V(H) \cap A = \frac{V(F) \cap T}{V(F) \cap R} \cong \frac{V(F) \cap T/[TV^*F]}{V(F) \cap R/[TV^*F]} \ .$$

Thus the result holds. □

Corollary 3.2

Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group. If H is any other group with a $\mathcal{V}$-marginal subgroup A such that $H/A \cong G$, then $V(H) \cap A$ is isomorphic to a subgroup of $\mathcal{V}M(G)$.

Proof.

Using Lemma 3.1 , $V(H) \cap A$ is a homomorphic image of $\mathcal{V}M(G)$. By Lemma 2.2, since $\mathcal{V}$ is a Schur-Baer variety and G is finite, it implies that $\mathcal{V}M(G)$ is a finite abelian group. Now, by the property of finite abelian groups, $V(H) \cap A$ is isomorphic to a subgroup of $\mathcal{V}M(G)$. □

Let $\mathcal{W}$ be a variety defined by an outer commutator word $w = w(x_1, \ldots, x_r)$, and $\mathcal{V}$ be the centre by centre by $\mathcal{W}$ variety of groups. Clearly the variety $\mathcal{V}$ will be defined by the commutator word $[w, x_1, x_2]$. Throughout the rest of this section we always work with the above variety $\mathcal{V}$.

The following definition and proposition are very important in our investigation, in which the concept of the *tranfer* has been generalized to the above variety $\mathcal{V}$.

Definition 3.3

Let E be any group with a subgroup B of finite index n , and $\{t_1, \ldots, t_n\}$ be a

right transversal of B in E . Let $\mathcal{V}$ be the variety of groups as defined before, and

$$\pi : B \longrightarrow B/V(B)$$

$$b \longmapsto \overline{b}$$

be the natural homomorphism, where $V(B)$ is the verbal subgroup of B with respect to the variety $\mathcal{V}$.

Now, for all $y_1, \ldots, y_r$ in B and x in E we define the following map

$$\tau_{y_1,\ldots,y_r} = \tau : E \longrightarrow B/V(B) \qquad ,$$

given by $\quad x \longmapsto \prod_{i=1}^{n} [w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x t_{ix}^{-1})\pi]$, where $Bt_i x = Bt_{ix}$, for all $x \in E$.

The following result shows that the map τ has the property of being a generalized transfer of the group E .

Proposition 3.4

With the above set up, τ is a well-defined homomorphism and indepedent of the choice of the transversal set. Also, if x is marginal in E, then

$$(x)\tau = \overline{[w(y_1, \ldots, y_r), x^n]} \quad .$$

Proof.

Clearly τ is well-defined. Let $x, y \in E$, then we have $\quad t_i xy t_{ixy}^{-1} = t_i x t_{ix}^{-1} t_{ix} y t_{(ix)y}^{-1}$. By the form of the commutator word which defines the variety $\mathcal{V}$, modulo $V(B)$, we obtain

$$[w(a_1, \ldots, a_r), [a_{r+1}, a_{r+2}]] \equiv [w(a_1, \ldots, a_r), a_{r+1}, a_{r+2}] \equiv 1 \quad , \quad (*)$$

9

for all $a_i \in B$. Thus

$$(xy)\tau = \prod_{i=1}^{n}[w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x y t_{ixy}^{-1})\pi]$$

$$= \prod_{i=1}^{n}[w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x t_{ix}^{-1})\pi(t_{ix} y t_{(ix)y}^{-1})\pi]$$

$$\equiv \prod_{i=1}^{n}[w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x t_{ix}^{-1})\pi][w(\overline{y_1}, \ldots, \overline{y_r}), (t_j y t_{jy}^{-1})\pi] \qquad (mod \ V(B))$$

$$= (x)\tau(y)\tau \quad , \quad putting \ ix = y \ .$$

Hence τ is a homomorphism.

Now, let $\{t'_1, \ldots, t'_n\}$ be another transversal set of B in E, then $t'_i = b_i t_i$ for some $b_i \in B$, $1 \leq i \leq n$. This implies that, for all $x \in E$.

$$t'_i x t'_{ix}{}^{-1} = b_i t_i x t_{ix}^{-1} b_{ix}^{-1} \equiv b_i b_{ix}^{-1} t_i x t_{ix}^{-1} \quad (mod \ \gamma_2(B)) \ .$$

Now, using $(*)$ and the above fact, mod $V(B)$,we have

$$\prod_{i=1}^{n}[w(\overline{y_1}, \ldots, \overline{y_r}), (t'_i x t'_{ix}{}^{-1})\pi] \equiv [w(\overline{y_1}, \ldots, \overline{y_r}), \prod_{i=1}^{n}(t'_i x t'_{ix}{}^{-1})\pi]$$

$$\equiv [w(\overline{y_1}, \ldots, \overline{y_r}), \prod_{i=1}^{n}(t_i x t_{ix}^{-1})\pi] \qquad (since \ \prod_{i=1}^{n} b_i b_{ix}^{-1} = 1)$$

$$\equiv \prod_{i=1}^{n}[w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x t_{ix}^{-1})\pi] = (x)\tau,$$

which implies that τ is independent of the choice of the transversal set.

Clearly every elements x of E acts on the transversal set $\{t_1, \ldots, t_n\}$. Assume there are k orbits with the following action:

$$t_{i_1} \ , \ t_{i_2} = t_{i_1}x \ , \ \ldots, \ t_{i_{l_i}} = t_{i_1}x^{l_i - 1} \ , \quad and \ t_{i_1} = t_{i_1}x^{l_i} \ ,$$

10

where $1 \leq i \leq k$, $l_1 + \ldots + l_k = n$. Clearly

$$
t_{i_j} x = \begin{cases} t_{i_{j+1}} & , \ if \ \ j \neq l_i \\[2ex] t_{i_1} & , \ if \ \ j = l_i \end{cases}
$$

which implies that

$$
t_{i_j} x t_{i_j x}^{-1} = \begin{cases} 1 & , \ if \ \ j \neq l_i \\[2ex] t_{i_1} x^{l_i} t_{i_1}^{-1} & , \ if \ \ j = l_i \end{cases}
$$

Hence

$$
(x)\tau = \prod_{i=1}^{n} [w(\overline{y_1}, \ldots, \overline{y_r}), (t_i x t_{ix}^{-1})\pi]
$$

$$
= \prod_{i=1}^{k} [w(\overline{y_1}, \ldots, \overline{y_r}), (t_{i_1} x^{l_i} t_{i_1}^{-1})\pi] \ \ .
$$

Now, if x is marginal in E , using Turner-Smith's Theorem [10], and Witt's identity
we have

$$
1 = [w(x_1, \ldots, x_r), x_{r+1}, x] = [w(x_1, \ldots, x_r), x, x_{r+1}]
$$

$$
= [w(x_1, \ldots, x_r), [x, x_{r+1}]] \ \ , \ for \ all \ \ x_i \in E \ \ .
$$

So the following holds:

$$
(x)\tau = \prod_{i=1}^{k} [w(\overline{y_1}, \ldots, \overline{y_r}), (t_{i_1} x^{l_i} t_{i_1}^{-1})\pi]
$$

$$
= \overline{\prod_{i=1}^{k} [w(y_1, \ldots, y_r), t_{i_1} t_{i_1}^{-1} x^{l_i} [x^{l_i}, t_{i_1}^{-1}]]}
$$

$$
= \overline{\prod_{i=1}^{k} [w(y_1, \ldots, y_r), [x^{l_i}, t_{i_1}^{-1}]][w(y_1, \ldots, y_r), x^{l_i}]}
$$

$$
= \overline{\prod_{i=1}^{k} [w(y_1, \ldots, y_r), x^{l_i}]} \qquad (since \ \ x^{l_i} \in V^*(E))
$$

11

$$= \overline{[w(y_1, \ldots, y_r), \prod_{i=1}^{k} x^{l_i}]}$$

$$= \overline{[w(y_1, \ldots, y_r), x^n]} \quad . \quad \square$$

Lemma 3.5

Let B be a subgroup of a group E with finite index n, and $\mathcal{V}$ be the same variety of groups as before. If $x \in V^*(E) \cap V(E)$, then for all $y_1, \ldots, y_r \in B$

$$[w(y_1, \ldots, y_r), x^n] \in V(B) \ .$$

In particular, if $L \subseteq V^*(E) \cap V(E)$, then $[W(B), L^n] \subseteq V(B)$.

Proof.

Clearly for all $x_i \in E$,

$$([w(x_1, \ldots, x_r), x_{r+1}, x_{r+2}])\tau = [w(x_1\tau, \ldots, x_r\tau), x_{r+1}\tau, x_{r+2}\tau] = 1 \quad in \quad B/V(B) \ .$$

So, by Proposition 3.4 for all $x \in V^*(E) \cap V(E)$

$$1 = (x)\tau = [w(y_1, \ldots, y_r), x^n] \quad in \quad B/B(V) \ ,$$

which implies that $[w(y_1, \ldots, y_r), x^n] \in V(B)$, for all $y_i \in B$. This also implies that $[W(B), L^n] \subseteq V(B)$, for any subgroup L of $V^*(E) \cap V(E)$. $\square$

Now we are able to state and prove the main result of this section.

Theorem 3.6

Let $\mathcal{V}$ be the variety of groups defined by the commutator word $[w, x, y]$ as before, and G be a finite group with a $\mathcal{V}$-stem cover

$$1 \longrightarrow L \longrightarrow G^* \longrightarrow G \longrightarrow 1 \ .$$

If H is a subgroup of index n in G and $H \cong B/L$, for some subgroup B of G^* . Then $[W(B), \mathcal{V}M(G)^n]$ is isomorphic to a subgroup of $\mathcal{V}M(H)$.

Proof.

By the definition of $\mathcal{V}$-stem cover we have

$$L \cong \mathcal{V}M(G) \quad and \quad L \leq V(G^*) \cap V^*(G^*) .$$

Clearly $[G^* : B] = n$, and so by Proposition 3.4,

$$[W(B), L^n] \subseteq V(B) .$$

Clearly L is marginal in B and since the vatiety $\mathcal{V}$ has the Schur-Baer property,

Corollary 3.2 implies that $L \cap V(B)$ is isomorphic to a subgroup of $\mathcal{V}M(H)$. We also

have

$$[W(B), \mathcal{V}M(G)^n] \subseteq L \cap V(B) .$$

Hence $[W(B), \mathcal{V}M(G)^n]$ is isomorphic to a subgroup of $\mathcal{V}M(H)$, which completes

the proof. $\square$

Now the above theorem has the following important corollaries, which generalize

Jones-Wiegold's Theorem [3,4] and Schur's Theorem [4,9].

Corollary 3.7

Let G be a finite group and G_p be a Sylow p-subgroup of G . If G has a $\mathcal{V}$-stem

cover $1 \to L \to G^* \to G \to 1$, such that $G_p \cong B/L$. Then $[W(B), \mathcal{V}M(G)_p]$ is

isomorphic to a subgroup of $\mathcal{V}M(G_p)$.

Proof.

Let the group G be of order $p^\alpha n$, where $(p, n) = 1$, then $[G : G_p] = n$. Using

Lemma 2.2 , $\mathcal{V}M(G)$ has an order dividing a power of $|G| = p^\alpha n$, and so we have

$\mathcal{V}M(G)^n = \mathcal{V}M(G)_p$. Now the result follows from Theorem 3.6 . □

Corollary 3.8

By the assumptions and the notations of Theorem 3.6, let $\mathcal{V}M(G)^{[n]}$ be the set of all elements of $\mathcal{V}M(G)$ of order coprime to n , then $[W(B), \mathcal{V}M(G)^{[n]}]$ is isomorphic to a subgroup of $\mathcal{V}M(H)$.

Proof.

Clearly, if G is a group then $G^{[n]}$ is a subgroup of G^n . Now the result holds using Theorem 3.6 . □

Corollary 3.9

By the assumptions and the notations of Theorem 3.6 ,let d and e be the exponents of $[W(B), \mathcal{V}M(G)]$ and $\mathcal{V}M(H)$, respectively. Then d divides ne . In particular, d divides n whenever H is cyclic.

Proof.

By the assumption, we have

$$\mathcal{V}M(G) \cong L \leq V^*(G^*), \quad L \leq B, \quad and \quad \mathcal{V}M(G) \leq V^*(B) .$$

So by Turner-Smith's Theorem [10], it follows that

$$[W(B), \mathcal{V}M(G)^n] = [W(B), \mathcal{V}M(G)]^n .$$

Therefore, by Theorem 3.6 , $[W(B), \mathcal{V}M(G)]^{ne} \subseteq \mathcal{V}M(H)^e = 1$.

Hence d divides ne . □

4. SCHUR'S TYPE THEOREM

In 1904 , I.Schur [9] proved that if G is a finite group and G_p is a Sylow p-subgroup of G then the Sylow p-subgroup of the Schur-multiplier of G, i.e. $M(G)_p$, is isomorphic to a subgroup of the Schur-multiplier of its Sylow p-subgroup, $M(G_p)$.

In this section by using our result together with a theorem of Moghaddam [6] (see Theorem 2.3), we prove a theorem of Schur's type with respect to the vatiety of nilpotent groups of class at most c $(c \geq 1)$, $\mathcal{N}_c$.

Now assumming the notations of Theorem 2.3, we first prove the following technical lemmas.

Lemma 4.1

Let the groups G_1 and G_2 in Theorem 2.3 be the cyclic groups of orders q^m and p^n , respectively, where p, q are any prime numbers. Then $T(Z_{p^n}, Z_{q^m})_{c+1} = 1$, when $c + 1$ is a prime number, and $\hat{T}(Z_{p^n}, Z_{q^m})_4 = 1$.

Proof.

Clearly the tensor product $Z_{p^n} \otimes Z_{q^m} = Z_{(p^n, q^m)} = 1$, by the definition as in Theorem 2.3, it follows that $\hat{T}(Z_{p^n} \otimes Z_{q^m}) = 1$.

Now, if $c + 1$ is a prime number then $T(Z_{p^n}, Z_{q^m})_{c+1}$ is the direct sum of the following tensor products:

$$Z_{p^n} \otimes Z_{q^m} \otimes Z_{t_1} \otimes \cdots \otimes Z_{t_{c-1}} \quad ,$$

where $t_i \in \{p^n, q^m\}$. Clearly $Z_{p^n} \otimes Z_{q^m} = 1$ and so each direct summand is trivial. Hence $T(Z_{p^n}, Z_{q^m}) = 1$.

Lemma 4.2

Let $G = H_1 \times H_2$ be the direct product of two finite p and q-groups H_1 and H_2, respectively. Then $\mathcal{N}_c M(G)_p \cong \mathcal{N}_c M(G_p)$, when $c + 1$ is any prime number or $c + 1 = 4$.

Proof.

Clearly $\mathcal{N}_c M(G_p) = \mathcal{N} M(H_1)$. By Theorem 2.3, we have

$$\mathcal{N}_c M(H_1 \times H_2) \cong \mathcal{N}_c M(H_1) \oplus \mathcal{N}_c M(H_2) \oplus T(H_2, H_1)_{c+1} ,$$

where $c + 1$ is a prime number. If $c + 1 = 4$, then the last term was denoted by $\hat{T}(H_2, H_1)_4$. Clearly any finite abelian p-group is isomorphic to some direct sum of cyclic p-groups $Z_{p^{t_i}}$. Hence using Lemma 4.1 and the definitions of $T(H_2, H_1)_{c+1}$ and $\hat{T}(H_2, H_1)_4$, it follows that both of these are trivial, which gives

$$\mathcal{N}_c M(H_1 \times H_2) \cong \mathcal{N}_c M(H_1) \oplus \mathcal{N}_c M(H_2) ,$$

where $c + 1$ is 4 or any prime number. Now, by Lemma 2.2, $\mathcal{N}_c M(H_i)$ is a p-group and a q-group for $i = 1, 2$, respectively. Thus

$$\mathcal{N}_c M(G)_p \cong \mathcal{N}_c M(H_1) = \mathcal{N}_c M(G_p) \quad . \quad \square$$

Now we state and prove the main result of this section, which is of Schur's type.

Theorem 4.3

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c and G be a finite nilpotent group with a Sylow p-subgroup G_p , say. If $c + 1$ is any prime number or 4,

then

$$\mathcal{N}_c M(G)_p \cong \mathcal{N}_c M(G_p) \ .$$

Proof.

It is known that a finite nilpotent group is the direct product of it's Sylow p-subgroups. So

$$G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_r} \ .$$

Now the result follows by induction on r and applying Lemma 4.2 . $\square$

The following corollary gives a criterion for a finite nilpotent group to have trivial Baer-invariant with respect to the variety of nilpotent groups of class at most c $(c \geq 1)$.

Corollary 4.4

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c and G be a finite nilpotent group. If every Sylow p-subgroup of G has trivial Baer-invariant, then so has G .

Proof.

Clearly the variety $\mathcal{N}_c$ has Schur-Baer property, so by Lemma 2.2, $\mathcal{N}_c M(G)$ has order dividing a power of $|G|$. Now Theorem 4.3 gives the result.

5. A COUNTEREXAMPLE

In this final section by giving an example we show that the result obtained in the previous sections can not be further generalized. Indeed, take the symmetric group

$$S_3 = < x, y \mid x^2, y^3, xyxy > \ ,$$

which is isomorphic to the semidirect product of two cyclic groups of orders two and three. So we have the following free presentation

$$1 \longrightarrow R = < x^2, y^3, xyxy >^F \longrightarrow F = < x, y > \longrightarrow S_3 \longrightarrow 1 \quad . \qquad (*)$$

It is known, by [4] , that $M(S_3) = 1$, so by the definition of Schur-multiplier we have

$$R \cap \gamma_2(F) = [R, F]$$

Clearly the Bear-invariant of S_3 , with respect to the variety of nilpotent groups of class 2 , is as follows:

$$\mathcal{N}_2 M(S_3) \cong \frac{R \cap \gamma_3(F)}{[R, F, F]} \quad .$$

Now we shall prove that one of the following embeddings does not hold

$$(\mathcal{N}_2 M(S_3))_2 \hookrightarrow \mathcal{N}_2 M((S_3)_2) \qquad or \qquad (\mathcal{N}_2 M(S_3))_3 \hookrightarrow \mathcal{N}_2 M((S_3)_3) \quad .$$

clearly $\mathcal{N}_2 M((S_3)_2)$ and $\mathcal{N}_2 M((S_3)_3)$ are both trivial, so it is enough to show that either $\mathcal{N}_2 M(S_3)_2$ or $\mathcal{N}_2 M(S_3)_3$ is non-trivial. But, as $\mathcal{N}_2$ is a Schur-Baer variety, Lemma 2.2 implies that $|\mathcal{N}_2 M(S_3)|$ is a 6-number. Hence we only need to show that $\mathcal{N}_2 M(S_3)$ is non-trivial, which we deal with in the rest of the section.

The following technical lemmas shorten the proof of the main theorem. We keep the above assumption and notation throughout the rest of the paper.

The proof of the following lemmas is a routine commutator calculation.

Lemma 5.1

(i) $[y, x, y] \in R$, and $[y, x, y], [y, x, x, y] \in R \cap \gamma_2(F) = [R, F]$,

(ii) $[y, x, y, a_1, \ldots, a_n] \in [R, F, F]$, for all $a_i \in F$ and $n \in \mathbf{N}$,

(iii) $[y, x, x, y, b_1, \ldots, b_m] \in [R, F, F]$, for all $b_i \in F$ and $m \in \mathbf{N}$.

Lemma 5.2

The following identities hold, modulo

$$T_1 = < [y, x, y, a_1 \ldots, a_n], [y, x, x, y, b_1, \ldots, b_m] \mid a_i, b_j \in F, n, m \in \mathbf{N} >:$$

(i) $[x^2, y, y] \equiv [y, x, y]^{-2}[x, y, x, y]$

(ii) $[x^2, y, x] \equiv [x, y, x]^{[x,y]}[x, y, x, x]^{[x,y]}[x, y, x]$

(iii) $[y^3, x, y] \equiv [y, x, y]^3$

(iv) $[y^3, x, x] \equiv [y, x, x]^{[y,x]^2}[y, x, x]^{[y,x]}[y, x, x]$

(v) $[xyxy, x, y] \equiv [y, x, y]^2[y, x, x, y]$

(vi) $[xyxy, x, x] \equiv [y, x, x]^{[y,x]}[y, x, x, x]^{[y,x]}[y, x, x]$

(vii) $[xyxy, y, y] \equiv [y, x, y]^{-2}[x, y, x, y]$

(viii) $[xyxy, y, x] \equiv [x, y, x]^{[x,y]}[x, y, x, x]^{[x,y]}[x, y, x]$

Lemma 5.3

$$[y, x, y] \; , \; [x, y, x, y] \; , \; [y, x, x, y] \in [R, F, F] \quad .$$

Proof.

By Hall-Witt's identities we have

$$[[y, x], y, x]^{y^{-1}}[y^{-1}, x^{-1}, [y, x]]^x[x, [y, x]^{-1}, y^{-1}]^{[y,x]} = 1 \quad . \qquad (1)$$

By Lemma 5.1 (ii),

$$[[y,x],y,x]^{y^{-1}} \in [R,F,F] \ .$$

We also have,

$$[y^{-1}, x^{-1}, [y,x]] = [y^{-3}y^2, x^{-2}x, [y,x]] \equiv [y^2, x, [y,x]] \qquad (mod \ [R,F,F])$$

$$\equiv [y, x, [y,x]]^2 \equiv 1 \qquad (mod \ [R,F,F]) \ .$$

Therefore, by (1)

$$[y^{-1}, x^{-1}, [y,x]]^x \in [R,F,F] \quad \Longrightarrow \quad [x, [y,x]^{-1}, y^{-1}] \in [R,F,F] \ .$$

But

$$[x, [y,x]^{-1}, y^{-1}] \equiv [[x,y,x]^{-1}, y^{-1}] \equiv [[x,y,x]^{-1}, y^{-3}y^2]$$

$$\equiv [x, y, x, y^2]^{-1} \equiv [x, y, x, y]^{-2} \qquad (mod \ [R,F,F]) \ ,$$

hence

$$[x, y, x, y]^2 \in [R,F,F] \ .$$

Using Lemma 5.2 (i) , (ii) , $[x, y, x, y]^3 \in [R,F,F]$. Thus

$$[x, y, x, y] \in [R,F,F] \ .$$

Therefore by Lemma 5.2 (i) , (ii) , (v) , we have

$$[y, x, y] \quad and \quad [y, x, x, y] \in [R,F,F] \ . \ \square$$

The following lemma can be proved by using Lemmas 5.1,5.2, and 5.3 :

Lemma 5.4

Let

$$T_2 = <T_1 , \ [y,x,x,[f_1,f_2,f_3],a_1,\ldots,a_n] , \ [y,x,x,[f_1',f_2'],b_1,\ldots,b_m] , \ [y,x,y] ,$$

$$[y,x,x,y] , \ [x,y,x,y] \mid f_1,f_2,f_3,f_1',f_2',a_i,b_j \in F , \ n,m \in \mathbf{N} > .$$

Then the following identities hold, modulo T_2 :

(i) $[x^2,y,y] \equiv [y^3,x,y] \equiv [xyxy,x,y] \equiv [xyxy,y,y] \equiv 1$

(ii) $[x^2,y,x] \equiv [y,x,x]^{-2}[y,x,x,x]^{-1}[y,x,x,[y,x]]^3[y,x,x,x,[y,x]] \equiv [xyxy,y,x]$

(iii) $[y^3,x,x] \equiv [y,x,x]^3[y,x,x,[y,x]]^3$

(iv) $[xyxy,x,x] \equiv [y,x,x]^2[y,x,x,x][y,x,x,[y,x]][y,x,x,x,[y,x]]$

Lemma 5.5

Let

$$T_3 = <T_2 , \ [y,x,x]^3 , \ [y,x,x,[y,x]]^3 , \ [y,x,x,[y,x]]^2[y,x,x,x,[y,x]] ,$$

$$[y,x,x]^2[y,x,x,x][y,x,x,[y,x]]^{-1} , \ [y,x,x,a_1,\ldots,a_r]^3 ,$$

$$[y,x,x,b_1,\ldots,b_s]^2[y,x,x,x,c_1,\ldots,c_t] , \ [y,x,x,x,[y,x],d_1,\ldots,d_u] ,$$

$$\mid a_i,b_j,c_k,d_l \in F , \ r,s,t,u \in \mathbf{N} > .$$

Then $T_3^F = T_3$ and $[R,F,F] \le T_3$.

Proof.

By the definitions of T_2 and T_3 , one can easily show that $T_3^F = T_3$. By Lemma 5.4 , every generator of $[R,F,F]$ is in T_3 , hence

$$[R,F,F] \le T_3 . \quad \square$$

Now we are in a position to prove the main theorem of this section.

Theorem 5.6

$$\mathcal{N}_2 M(S_3) \neq 1 \quad .$$

Proof.

Clearly the identity map $id : R \cap \gamma_3(F) \longrightarrow R \cap \gamma_3(F)$ is a homomorphism, and by Lemma 5.5, $id([R, F, F]) \leq T_3$. Also by $(*)$, we have

$$T_3 \leq R \cap \gamma_3(F) \quad .$$

It follows that the induced map

$$id^* : \frac{R \cap \gamma_3(F)}{[R, F, F]} \longrightarrow \frac{R \cap \gamma_3(F)}{T_3}$$

is also a homomorphism. By Lemma 5.5, $[y, x, x, [y, x]] \notin T_3$, but $[y, x, x, [y, x]] \in R \cap \gamma_3(F)$ so

$$id^*(\overline{[y, x, x, [y, x]]}) \neq 1$$

Thus

$$1 \neq \overline{[y, x, x, [y, x]]} \in \frac{R \cap \gamma_3(F)}{[R, F, F]} \cong \mathcal{N}_2 M(S_3) \quad .$$

Hence

$$\mathcal{N}_2 M(S_3) \neq 1 \quad . \quad \square$$

Now the following corollary gives the assertion of our counterexample.

Corollary 5.7

$$(\mathcal{N}_2 M(S_3))_2 \nrightarrow \mathcal{N}_2 M((S_3)_2)$$

or

$$(\mathcal{N}_2 M(S_3))_3 \nrightarrow \mathcal{N}_2 M((S_3)_3) \quad .$$

Proof.

Clearly $\mathcal{N}_2 M((S_3)_2) = 1 = \mathcal{N}_2 M((S_3)_3)$. By Theorem 5.6

$$(\mathcal{N}_2 M(S_3))_2 \neq 1 \quad or \quad (\mathcal{N}_2 M(S_3))_3 \neq 1 \quad .$$

Hence the result holds. □

Remark

Clearly S_3 is a non-nilpotent solvable group, so the above corollary shows that the results of section 4 can not be held for solvable groups, in general. Also it shows that the Schur's Theorem [4,9] and Jones-Wiegold's Theorem [3,4] can not be held for solvable groups with respect to the variety of nilpotent groups of class c , $c \geq 2$.

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